Journal of Engineering Mathematics **35:** 59–70, 1999.

On the stability and nonlinear dynamics of ocean-like wave systems with energy continuously distributed in direction

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Received 21 August 1997; accepted in revised form 11 June 1998

Abstract. Stability and dynamics of a distributed gravity traveling-wave system which is of practical importance in ocean-wave modeling are studied in this paper. A new cubic nonlinear evolution equation for the distributed wave system has been derived. The instability findings of the primary wave system are of critical value in the study of real wind waves and are technically a generalization of the Benjamin–Feir instability for a monochromatic water-wave system.

Keywords: ocean waves, wave stability, Benjamin–Feir instability, directional waves

1. Introduction

Recent low-grazing-angle radar observations (Werle [1]) suggest the general existence of wind waves propagating as multiple wave groups. The characteristics of the groups suggest that they may result from wave instability of the Benjamin–Feir [2] type. The radar is believed to be highly useful in its ability to see the wave groups, and groups have been seen propagating over a variety of angles, notably in the wind direction. These observations immediately give rise to the question as to whether infinitesimal waves distributed in their direction of propagation about the wind, constitute an unstable wave system in a sense similar to a monochromatic, unidirectional wave undergoing Benjamin–Feir instability. This question has never been raised in the literature, and therefore the present theoretical study of the instability of a wave system whose energy is continuously distributed was undertaken; in particular, the wave energy is assumed distributed on a straight line segment, bisected by the wind direction, in vector wave-number space. Therefore $\mathbf{k} = (p, q)$ is discrete in the wind direction, $p = p_0$, but distributed transversely over $-q_0 \leqslant q \leqslant q_0$, where we restricted our attention to $q_0/p_0 < 2^{-\frac{1}{2}}$.

In what follows, the Hamiltonian treatment of Zakharov [3], first formulated in 1968, is applied to derive a cubic nonlinear evolution equation appropriate to this particular distributed wave system, an integro-partial differential equation, and linear stability of the wave system is subsequently studied.

2. Hamiltonian formulation

The problem is first formulated in wave-number space, and subsequently partially transformed to physical space. With the usual nomenclature,

$$\eta(\mathbf{k}, t) = \frac{1}{2\pi}\int \eta(\mathbf{x}, t)\,\mathrm{e}^{-i\mathbf{k}.\mathbf{x}}\,\mathrm{d}\mathbf{x}, \quad \psi(\mathbf{k}, t) = \frac{1}{2\pi}\int \psi(\mathbf{x}, t)\,\mathrm{e}^{-i\mathbf{k}.\mathbf{x}}\,\mathrm{d}\mathbf{x}, \tag{1}$$

where $\eta(\mathbf{x}, t)$ is the wave elevation and $\psi(\mathbf{x}, t)$ the potential on the water surface, the equations of the system in Hamiltonian form become

$$\frac{\partial \eta_k}{\partial t} = \frac{\delta H}{\delta \psi_k^*}, \qquad \frac{\partial \psi_k}{\partial t} = -\frac{\delta H}{\delta \eta_k^*}, \tag{2}$$

where $H[\eta(\mathbf{k}, t), \psi(\mathbf{k}, t)]$ has been given to fourth order in the wave slope (Zakharov [4]). According to the procedure followed by Zakharov [4] and followed up by Krasitskii [5], we may first introduce a normal complex variable, $a(\mathbf{k}, t) = 2^{-\frac{1}{2}}(\gamma_k \eta(\mathbf{k}, t) + i\psi(\mathbf{k}, t)/\gamma_k)$, followed by its canonical transformation to another normal variable, b_k, allowing the expression of the governing equation in a particularly useful form,

$$i\frac{\partial b_k}{\partial t} = \frac{\delta H}{\delta b_k^*} = \omega_k b_k + \int T(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) b_{k1}^* b_{k2} b_{k3} \delta_{k+k1-k2-k3}\, \mathrm{d}\mathbf{k}_1\, \mathrm{d}\mathbf{k}_2\, \mathrm{d}\mathbf{k}_3 \tag{3}$$

or

$$i\frac{\partial B_k}{\partial t} = \int T(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) B_{k1}^* B_{k2} B_{k3}\, \mathrm{e}^{i[\omega(k)+\omega(k1)-\omega(k2)-\omega(k3)]t} \times \delta_{k+k1-k2-k3}\, \mathrm{d}\mathbf{k}_1\, \mathrm{d}\mathbf{k}_2\, \mathrm{d}\mathbf{k}_3, \tag{4}$$

where $B_k = b_k\, \mathrm{e}^{i\omega(k)t}$; the interaction kernel T and others are given in the Appendix.

As is well known, this Zakharov equation (4) governs the weakly nonlinear dynamics of surface water waves, and is completely equivalent to the full water-wave equations in physical space, to third order.

3. Model equations

A general treatment of (4) does not exist, so we first specialize it to the case of interest. Our concern is the perturbation and stability of the primary wave system which has a straight line segment in the wave number space bisected by the wind direction. That is, $k = (p_0, q)$, where $-q_0 \leqslant q \leqslant q_0 (q_0/p_0 < 2^{-\frac{1}{2}})$. Therefore, we consider the wave number space: $\mathbf{k} = (p, q)$, where $p_0 - \delta p \leqslant p \leqslant p_0 + \delta p (\delta p/p_0$ small), and $-q_0 \leqslant q \leqslant q_0$. Here p_0 and q_0 are arbitrary initially, except for the restriction on q_0/p_0.

Having restricted the range of $\mathbf{k}$, we may correspondingly simplify the evolution equation (3) by approximating ω_k and the kernel T near $p = p_0$. We first expand the linear dispersion relation around $p = p_0$:

$$\omega_k \equiv \omega(p, q) \cong \omega_0 + \omega_1(p - p_0) + \tfrac{1}{2}\omega_2(p - p_0)^2, \tag{5}$$

where coefficients ω_j are given in the Appendix and take

$$T(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \cong T[(p_0, q), (p_0, q_1), (p_0, q_2), (p_0, q_3)] \equiv T_0(q, q_1, q_2, q_3). \tag{6}$$

Then (3) becomes,

$$i\frac{\partial b(p, q, t)}{\partial t} = [\omega_0(q) + \omega_1(q)\cdot(p - p_0) + \tfrac{1}{2}\omega_2(q)\cdot(p - p_0)^2] b(p, q, t) + \int T_0(q, q_1, q_2, q_3) b^*(p_1, q_1) b(p_2, q_2) b(p_3, q_3) \times \delta_{\mathbf{k}+\mathbf{k}1-\mathbf{k}2-\mathbf{k}3}\, \mathrm{d}p_1\, \mathrm{d}q_1\, \mathrm{d}p_2\, \mathrm{d}q_2\, \mathrm{d}p_3\, \mathrm{d}q_3. \tag{7}$$

To return to x-space, we transform (7) with respect to $(p - p_0)$, using

$$b(x,q,t) = \int b(p,q,t)\, \mathrm{e}^{i[p-po]x}\, \mathrm{d}p \tag{8}$$

and obtain

$$\begin{aligned} i\frac{\partial b(x,q,t)}{\partial t} &= \omega_0(q) b(x,q,t) - i\omega_1(q)\frac{\partial b(x,q,t)}{\partial x} - \tfrac{1}{2}\omega_2(q)\frac{\partial^2 b(x,q,t)}{\partial x^2} \\ &\quad + \int T_0(q,q_1,q_2,q_3) b^*(x,q_1,t) b(x,q_2,t) b(x,q_3,t) \\ &\quad \times \delta_{q+q1-q2-q3}\, \mathrm{d}q_1\, \mathrm{d}q_2\, \mathrm{d}q_3. \end{aligned} \tag{9}$$

From the above context there should be no confusion of $b(x,q,t)$ with $b(p,q,t)$.

We can put Equation (9) in the following form by writing,

$$b(x,q,t) = B(x,q,t)\, \mathrm{e}^{-i\omega o(q)t},$$

and, using a moving reference, $x - \omega_1 t \to x$, we have

$$\begin{aligned} i\frac{\partial B_q}{\partial t} &= -\tfrac{1}{2}\omega_2(q)\frac{\partial^2 B_q}{\partial x^2} \\ &\quad + \int T_0(q,q_1,q_2,q_3) B^*_{q1} B_{q2} B_{q3}\, \mathrm{e}^{i[\omega(q)+\omega(q1)-\omega(q2)-\omega(q3)]t} \\ &\quad \times \delta_{q+q1-q2-q3}\, \mathrm{d}q_1\, \mathrm{d}q_2\, \mathrm{d}q_3. \end{aligned} \tag{10}$$

Assuming symmetry about $q = 0$, corresponding physically to the case of the resonant wave pairs taking a special configuration in wave number plane during resonant wave interaction, a parallelogram, such that $(q_2, q_3) = (q, q_1)$, we have,

$$i\frac{\partial B_q}{\partial t} = -\tfrac{1}{2}\omega_2(q)\frac{\partial^2 B_q}{\partial x^2} + 2q_0 B_q \int T_0(q,q_1)|B_{q1}|^2\, \mathrm{d}q_1, \tag{11}$$

where $\omega_2(q)$ is given in (A12), $T_0(q,q_1) \equiv T_0(q,q_1,q,q_1)$ is real and symmetric, and its integration is over $(-q_0, q_0)$.

Equation (11) now governs the weakly nonlinear dynamics of a p-narrow banded and q-distributed, q-symmetric water wave system in $\mathbf{k} = (p,q)$ and is therefore a generalization of the cubic-nonlinear Schrödinger equation for a unidirectional narrow-banded water wave system.

4. Primary wave system and its instability

Equation (11) admits a p-unmodulated quasi-stationary wave envelope solution, analogous to the Stokes wave of the cubic-nonlinear Schrödinger equation,

$$B_0(x,q,t) = \alpha(q)\, \mathrm{e}^{-i\beta(q)t}, \tag{12}$$

where $\beta(q) = 2q_s \int T_0(q, q_1)|\alpha(q_1)|^2 \,\mathrm{d}q_1$, integrated over $(-q_0, q_0)$. This quasi-stationary envelope solution corresponds to the primary wave system with a p-discrete $(p = p_0)$ and transversely distributed spectrum $(q \in [-q_0, q_0])$.

We shall be concerned in the following with the stability of the primary wave system. We first perturb the primary wave system, $B_0(q,t) = \alpha(q)\,\mathrm{e}^{-i\beta(q)t}$,

$$B(x,q,t) = B_0(q,t)[1 + \delta B(x,q,t)], \tag{13}$$

where δB is a small but otherwise arbitrary perturbation, and linearize it with respect to $\delta B(x,q,t)$,

$$i\frac{\partial \delta B_q}{\partial t} = -\tfrac{1}{2}\omega_2(q)\frac{\partial^2 \delta B_q}{\partial x^2} + 2q_0 \int T_0(q, q_1)|\alpha(q_1)|^2(\delta B_{q1} + \delta B^*_{q1})\,\mathrm{d}q_1, \tag{14}$$

where $\delta B_q \equiv \delta B(x,q,t)$.

The general solution to the perturbation equation (14) can be written in the form

$$\delta B(x,q,t) = \delta B^+(q)\,\mathrm{e}^{i(Kx+\Omega t)} + [\delta B^-(q)\,\mathrm{e}^{i(Kx+\Omega t)}]^*, \tag{15}$$

where the eigen number K is real, the eigenvalue Ω can be complex, and $\delta B^+(q) \neq \delta B^-(q)$ in general. We then have the eigenvalue problem for $\{\delta B^+, \delta B^-\}$,

$$\begin{pmatrix} \Omega + \frac{1}{2}K^2\omega_2 + L_\alpha & L_\alpha \\ -L_\alpha & \Omega + \frac{1}{2}K^2\omega_2 + L_\alpha \end{pmatrix} \cdot \begin{pmatrix} \delta B^+ \\ \delta B^- \end{pmatrix} = 0, \tag{16}$$

where the integral operator L_α is defined by, $L_\alpha s(q) = 2q_0 \int T_0(q, q_1)|\alpha(q_1)|^2 s(q_1)\,\mathrm{d}q_1$.

Equivalently, we may write,

$$\begin{pmatrix} \Omega & \frac{1}{2}K^2\omega_2(q) \\ \frac{1}{2}K^2\omega_2(q) + 2|\alpha(q)|^2 L_0 & \Omega \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = 0, \tag{17}$$

where

$$u(q) = |\alpha(q)|^2[\delta B^+(q) + \delta B^-(q)], \quad v(q) = |\alpha(q)|^2[\delta B^+(q) - \delta B^-(q)] \tag{18}$$

and $L_0 s(q) = 2q_0 \int T_0(q, q_1)s(q_1)\,\mathrm{d}q_1$.

The eigenvalue problem (17) is difficult to solve, because of its varying coefficients. We shall approach the problem by studying its behavior near $\Omega^2 = 0$, using a perturbation method. From (17), we have

$$\{\Omega^2 - [\tfrac{1}{2}K^2\omega_2(q)]^2 - K^2\omega_2(q)|\alpha(q)|^2 L_0\}u(q) = 0. \tag{19}$$

For a given perturbation δB, Equation (17) defines a dispersion relation, $\Omega^2 = f(K^2)$, with $(K^2, \Omega^2) = (0, 0)$ being always a solution. For reasons of stability, we are interested in nontrivial zero solutions. That is, $\Omega^2 = 0$, for some nontrivial values of $\mathrm{K}^2_{nt} \neq 0$. If such solutions exist, $(K^2, \Omega^2) = (K^2_{nt}, 0)$, we may immediately deduce the stability of the solutions nearby (stable if $\Omega^2 > 0$ always or unstable if $\Omega^2 < 0$ somewhere). To this end, we shall transform (19) to a form convenient to cope with

$$\tfrac{1}{4}K^2 w(q) - \int T_N(q, q_1) w(q_1)\,\mathrm{d}q_1 = 0, \tag{20}$$

where

$$w(q) = u(q)|\alpha(q)|^{-1}[-\omega_2(q)]^{\frac{1}{2}} \left\{ \frac{1-\Omega^2}{[\frac{1}{2}K^2\omega_2(q)]^2} \right\}^{\frac{1}{2}}. \tag{21}$$

The kernel $T_N(q, q_1)$ is symmetric,

$$\begin{aligned} T_N(q, q_1) \;=\; & 2q_0 T_0(q, q_1)|\alpha(q)||\alpha(q_1)|[-\omega_2(q)]^{-\frac{1}{2}}[-\omega_2(q_1)]^{-\frac{1}{2}} \\ & \times \left\{ \frac{1-\Omega^2}{[\frac{1}{2}K^2\omega_2(q)]^2} \right\}^{-\frac{1}{2}} \left\{ \frac{1-\Omega^2}{[\frac{1}{2}K^2\omega_2(q_1)]^2} \right\}^{-\frac{1}{2}}. \end{aligned} \tag{22}$$

Assuming that $(q_0/p_0)^2 < \frac{1}{2}$ (note that $\omega_2(q) < 0$ for $(q/p_0)^2 \leqslant (q_0/p_0)^2 < \frac{1}{2}$), and Ω^2 small (more precisely $\Omega^2/[\frac{1}{2}K^2\omega_2(q)]^2 \ll 1$), we have

$$\begin{aligned} T_N(q, q_1) \;\cong\; & 2q_0 T_0(q, q_1)|\alpha(q)||\alpha(q_1)|[-\omega_2(q)]^{-\frac{1}{2}}[-\omega_2(q_1)]^{-\frac{1}{2}} \\ & \times\{1 + 2\Omega^2 K^{-4}[\omega_2(q)]^{-2} + 2\Omega^2 K^{-4}[\omega_2(q_1)]^{-2}\}. \end{aligned} \tag{23}$$

Thus (20) becomes,

$$[\tfrac{1}{4}K^2 - L_1 - 2\Omega^2 K^{-4} L_2]w = 0, \tag{24}$$

where L_j are given by,

$$\begin{aligned} L_1 s(q) \;=\; & 2q_0 \int T_0(q, q_1)|\alpha(q)||\alpha(q_1)|[-\omega_2(q)]^{-\frac{1}{2}}[-\omega_2(q_1)]^{-\frac{1}{2}} s(q_1)\,\mathrm{d}q_1, \\ L_2 s(q) \;=\; & 2q_0 \int T_0(q, q_1)|\alpha(q)||\alpha(q_1)|[\omega_2(q)\omega_2(q_1)]^{-\frac{1}{2}} \\ & \times\{[\omega_2(q)]^{-2} + [\omega_2(q_1)]^{-2}\} s(q_1)\,\mathrm{d}q_1. \end{aligned} \tag{25}$$

We can now solve Equation (24) by using a prescribed perturbation method. Starting with certain unperturbed solution, $\Omega^2 = 0$, $K^2 \neq 0$, we obtain from (24), $\frac{1}{4}K^2 w = L_1 w$. It is easy to show that L_1 is self-adjoint and compact (the kernel of L_1 is real, symmetric, and bounded, when $|\alpha(q)|$ is assumed continuous) and by the Hilbert–Schmidt expansion theory there therefore exists a set of non-empty, real eigenvalues $\{\lambda_j\}$ satisfying

$$\lambda_1^- < \lambda_2^- < \cdots < \lambda_N^- < \cdots < 0 < \cdots < \lambda_N^+ < \cdots < \lambda_2^+ < \lambda_1^+,$$

where $\lambda_1^+ = \sup_{s(q)\neq 0}\{\int L_1 s(q) s^*(q_1)\,\mathrm{d}q_1 / \int s(q) s^*(q_1)\,\mathrm{d}q_1\}$, and eigenfunctions $\{w_j\}$ satisfy, $\lambda_j^\pm w_j^\pm = L_1 w_j^\pm$, for $j = 1, 2$. We have thus a set of nontrivial zeros $\Omega^2 = 0$ at $K_j^2 = 4\lambda_j^+$, for $j = 1, 2$. Below we will show that $\Omega^2(K^2) < 0$ for some $K_j^2 < K^2 < K_{j+1}^2$.

By expanding (24) at $(\Omega^2, K^2, w) = (0, K_1^2, w_1)$,

$$\Omega^2 = \delta\Omega + \cdots, \quad K^2 = K_1^2 + \delta K + \cdots, \quad w = w_1 + \delta w + \cdots \tag{26}$$

we have,

$$\begin{aligned} &\tfrac{1}{4}K_1^2 w_1 - L_1 w_1 = 0 \\ &\tfrac{1}{4}K_1^2 \delta w - L_1 \delta w = 2\delta\Omega K_1^{-4} L_2 w_1 - \tfrac{1}{4}\delta K w_1. \end{aligned} \tag{27}$$

The solvability condition for the second equation of (27) requires that its right-hand-side be orthogonal to the adjoint solutions of the first equation:

$$(\delta\Omega/\delta K)\langle L_2 w_1, w_1^*\rangle = \tfrac{1}{8}K_1^4 \langle w_1, w_1^*\rangle \equiv \tfrac{1}{8}K_1^4 ||w_1||^2. \tag{28}$$

It is easy to show from definition (25) and the symmetry of L_2 that the inner product $\langle L_2 w_1, w_1^*\rangle$ is always positive, and hence we conclude from (28), $\delta\Omega/\delta K > 0$. Thus, for small $\delta K < 0$, we must have, $\delta\Omega < 0$. This demonstrates the existence of an instability near $(\Omega^2, K^2) = (0, K_1^2)$: $\Omega^2 < 0$ for $K_2^2 < K^2 < K_1^2$. By the same token and the continuity of $\Omega^2(K^2)$, it can be further deduced that there exist an accountable infinite number of instabilities within the alternating intervals $(K_{2n}^2, K_{2n-1}^2)(n = 1, 2)$ with decreasing growth rates. It is interesting to note the similarity of this band of instabilities to the one in Floquet's theory for a linear periodic system. For a given continuous wave distribution, $\alpha = \alpha(q)$, the instabilities and associated growth rates can readily be carried out numerically.

To give an explicit example, we consider a primary wave system uniformly distributed on the segment, $|\alpha(q)| =$ constant. In this case, we first compute the following eigenvalue problem,

$$\lambda w = L_1 w. \tag{29}$$

In numerical computation, Equation (29) needs to be non-dimensionalized: $\lambda/p_0^2 \to \lambda$, $q/p_0 \to q$. To compute the eigenvalues (λ_1^+ in particular) of (29), a modified quadrature method is first used, and the results are then double-checked by means of the Rayleigh–Ritz method. Since these numerical methods are well known (Baker [6]), only the numerical results need to be given here. It should be noted that the instability interval spanned by $(0, K_1^2)$ in the (Ω^2, K^2) plane actually corresponds to $(1 - K_1, 1 + K_1)$ in the non-dimensionalized p-plane, and K_1^2 is a function of $q_0 (0 < q_0 < 2^{-\frac{1}{2}})$.

Recall that the primary wave system is distributed on the interval, $-q_0 \leqslant q \leqslant q_0$, and it can readily be shown that the primary wave system degenerates to the discrete Benjamin–Feir system in the limit, $q_0 \to 0$, with $K_j^2 = 0$ for $j = 2, 3$. In the other limit, $q_0 \to 2^{-\frac{1}{2}}$, $\omega_2(q_0)$ approaches zero and the kernel of L_1 becomes singular. It is interesting to note that the limiting wedge-shaped region formed by the wave number vectors $(p, q) = (1, \pm 2^{-\frac{1}{2}})$ has a semi-angle, $\tan^{-1}(2^{-\frac{1}{2}}) = 35{\cdot}3^0$, exactly as in the case of the Kelvin ship wave pattern (Whitham [7]). A further study of this singularity $(p, q) = (1, 2^{-\frac{1}{2}})$ is thus of both theoretical interest and practical importance.

$$\begin{array}{llll} q_0 = 0{\cdot}0 & 0{\cdot}3535 & 0{\cdot}53 & 0{\cdot}70 \\ K_1^2 = 1{\cdot}000 & 1{\cdot}201 & 1{\cdot}649 & 9{\cdot}236 \\ K_2^2 = 0{\cdot}000 & 0{\cdot}090 & 0{\cdot}306 & 4{\cdot}670. \end{array} \tag{30}$$

The physical interpretation of (30) requires the following identity,

$$\delta p_j / p_0 = (2^{-\frac{1}{2}})(q_0/p_0)|\alpha|(\omega_0^{-\frac{1}{2}} p_0^{\frac{5}{2}})K_j, \tag{31}$$

where the one-dimensional spectral energy density is $2\omega_k|\alpha|^2/p_0$, and $q_0|\alpha|$ is assumed constant in this illustrative example.

5. Concluding remarks

In this paper, we have established a nonlinear dynamical model, Equation (11), for a distributed water-wave system, which is of practical importance in wind-wave modeling, and based on this model we have demonstrated the instability of certain primary continuously distributed wave systems. We have shown the primary wave system with a narrow spectral distribution on (p_0, q), where $q \in [-q_0, q_0]$ and $q_0/p_0 < 2^{\frac{1}{2}}$, to be unstable to arbitrary small perturbations, disregarding any specific form of its distribution over (p_0, q), which is a key extension of the result for a classic Stokes-wave system. In fact, when q_0 shrinks to zero, the result of Benjamin–Feir for a Stokes wave is naturally recovered.

Knowledge of the growth rate of specific sidebands, a function of $q < q_0$ and $|\alpha|$ requires the calculation of the complete set of growth curves associated with the pairs (K_j, K_{j+1}), j odd.

Those p-discrete sidebands, Equation (15), of maximum growth rate can be expected to grow under wind forcing and evolve further through energy transfer to a still lower side band and down shifting due to wave breaking (Tulin [8]), wave groups being produced in the process.

It now seems conceivable, and further study is required to confirm this, that in the ocean a finite number of these p-discrete modulated waves, each propagating within a different sector, can be brought about through the instability mechanism shown here in the case of the sector bisected by the wind direction.

Appendix

To fourth order, the Hamiltonian in (2) takes the form,

$$\begin{aligned} H &= H[\eta(\mathbf{k}, t), \psi(\mathbf{k}, t)] \\ &= \frac{1}{2}\int |\mathbf{k}|\psi_k\psi_k^*\,\mathrm{d}\mathbf{k} + \int K^{(3)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\psi_k\psi_{k1}\eta_{k2}\delta_{k+k1+k2}\,\mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2 \\ &\quad + \int K^{(4)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\psi_k\psi_{k1}\eta_{k2}\eta_{k2}\delta_{k+k1+k2+k3}\,\mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2\,\mathrm{d}\mathbf{k}_3 \\ &\quad + \frac{1}{2}\int g\eta_k\eta_k^*\,\mathrm{d}\mathbf{k}, \end{aligned} \tag{A1}$$

where

$$K^{(3)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = -\frac{1}{4\pi}[\mathbf{k}\cdot\mathbf{k}_1 + |\mathbf{k}|\cdot|\mathbf{k}_1|],$$

$$K^{(4)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \tag{A2}$$

$$= -\frac{1}{32\pi^2}(|\mathbf{k}| \cdot |\mathbf{k}_1|)[2|\mathbf{k}| + 2|\mathbf{k}_1| - |\mathbf{k} + \mathbf{k}_2| - |\mathbf{k} + \mathbf{k}_3| - |\mathbf{k}_1 + \mathbf{k}_2| - |\mathbf{k}_1 + \mathbf{k}_3|]$$

and the constant density has been set to unity without loss of generality.

We may combine Equations (2) into a single equation by introducing a pair of normal complex variables, $\{a(\mathbf{k}, t), a^*(\mathbf{k}, t)\}$,

$$a(\mathbf{k}, t) = (1/\sqrt{2})(\gamma_k \eta(\mathbf{k}, t) + i\psi(\mathbf{k}, t)/\gamma_k), \tag{A3}$$

where the dimensional factor, $\gamma_k = (\omega_k/|\mathbf{k}|)^{\frac{1}{2}}$ with $\omega_k = (g|\mathbf{k}|)^{\frac{1}{2}}$, has been chosen such that the terms $(\gamma_k \eta_k, \psi_k/\gamma_k)$ have the same dimensionality and, furthermore, the linearized Hamiltonian is diagonalized. The Hamiltonian Equations (2) then take the form

$$\begin{aligned}
i\frac{\partial a_k}{\partial t} &= \frac{\delta H}{\delta a_k^*} \\
&= \omega_k a_k + \int U^{(1)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) a_{k1} a_{k2} \delta_{k-k1-k2}\, \mathrm{d}\mathbf{k}_1\, \mathrm{d}\mathbf{k}_2 \\
&\quad + 2\int U^{(1)}(\mathbf{k}_2, \mathbf{k}_1, \mathbf{k}) a_{k1}^* a_{k2} \delta_{k+k1-k2}\, \mathrm{d}\mathbf{k}_1\, \mathrm{d}\mathbf{k}_2 \\
&\quad + \int U^{(3)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) a_{k1}^* a_{k2}^* \delta_{k+k1+k2}\, \mathrm{d}\mathbf{k}_1\, \mathrm{d}\mathbf{k}_2 \\
&\quad + \int V^{(1)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) a_{k1} a_{k2} a_{k3} \delta_{k-k1-k2-k3}\, \mathrm{d}\mathbf{k}_1\, \mathrm{d}\mathbf{k}_2\, \mathrm{d}\mathbf{k}_3 \\
&\quad + 3\int V^{(1)}(\mathbf{k}_3, \mathbf{k}_2, \mathbf{k}_1, \mathbf{k}) a_{k1}^* a_{k2}^* a_{k3} \delta_{k+k1+k2-k3}\, \mathrm{d}\mathbf{k}_1\, \mathrm{d}\mathbf{k}_2\, \mathrm{d}\mathbf{k}_3 \\
&\quad + \int V^{(2)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) a_{k1}^* a_{k2} a_{k3} \delta_{k+k1-k2-k3}\, \mathrm{d}\mathbf{k}_1\, \mathrm{d}\mathbf{k}_2\, \mathrm{d}\mathbf{k}_3 \\
&\quad + \int V^{(4)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) a_{k1}^* a_{k2}^* a_{k3}^* \delta_{k+k1+k2+k3}\, \mathrm{d}\mathbf{k}_1\, \mathrm{d}\mathbf{k}_2\, \mathrm{d}\mathbf{k}_3
\end{aligned} \tag{A4}$$

with the corresponding Hamiltonian,

$$\begin{aligned}
H &= \int \omega_k a_k a_k^*\, \mathrm{d}\mathbf{k} + \int U^{(1)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)(a_k^* a_{k1} a_{k2} + a_k a_{k1}^* a_{k2}^*) \delta_{k-k1-k2}\, \mathrm{d}\mathbf{k}\, \mathrm{d}\mathbf{k}_1\, \mathrm{d}\mathbf{k}_2 \\
&\quad + \frac{1}{3}\int U^{(3)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)(a_k^* a_{k1}^* a_{k2}^* + a_k a_{k1} a_{k2}) \delta_{k+k1+k2}\, \mathrm{d}\mathbf{k}\, \mathrm{d}\mathbf{k}_1\, \mathrm{d}\mathbf{k}_2 \\
&\quad + \int V^{(1)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)(a_k^* a_{k1} a_{k2} a_{k3} + a_k a_{k1}^* a_{k2}^* a_{k3}^*) \delta_{k-k1-k2-k3}\, \mathrm{d}\mathbf{k}\, \mathrm{d}\mathbf{k}_1\, \mathrm{d}\mathbf{k}_2\, \mathrm{d}\mathbf{k}_3
\end{aligned}$$

$$+\frac{1}{2}\int V^{(2)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)(a_k^* a_{k1}^* a_{k2} a_{k3})\delta_{k+k1-k2-k3}\,\mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2\,\mathrm{d}\mathbf{k}_3$$

$$+\frac{1}{4}\int V^{(4)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$$

$$\times(a_k^* a_{k1}^* a_{k2}^* a_{k3}^* + a_k a_{k1} a_{k2} a_{k3})\delta_{k+k1+k2+k3}\,\mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2\,\mathrm{d}\mathbf{k}_3, \qquad \text{(A5)}$$

where

$$U^{(1)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) = -U(-\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) - U(-\mathbf{k},\mathbf{k}_2,\mathbf{k}_1) + U(\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}),$$

$$U^{(3)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) = U(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) + U(\mathbf{k},\mathbf{k}_2,\mathbf{k}_1) + U(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})$$

with $U(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) = \frac{1}{8\pi}(\omega_k/|\mathbf{k}|)^{\frac{1}{2}}(\omega_{k1}/|\mathbf{k}_1|)^{\frac{1}{2}}(|\mathbf{k}_2|/\omega_{k2})^{\frac{1}{2}}[\mathbf{k}\cdot\mathbf{k}_1 + |\mathbf{k}|\cdot|\mathbf{k}_1|]$,

$$\begin{aligned}
V^{(1)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) &= \tfrac{1}{3}[-V(-\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) - V(-\mathbf{k},\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_3)\\
&\quad -V(-\mathbf{k},\mathbf{k}_3,\mathbf{k}_1,\mathbf{k}_2) + V(\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k},\mathbf{k}_3)\\
&\quad +V(\mathbf{k}_1,\mathbf{k}_3,-\mathbf{k},\mathbf{k}_2) + V(\mathbf{k}_2,\mathbf{k}_3,-\mathbf{k},\mathbf{k}_1)],\\
V^{(2)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) &= [V(-\mathbf{k},-\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) + V(\mathbf{k}_2,\mathbf{k}_3,-\mathbf{k},-\mathbf{k}_1)\\
&\quad -V(-\mathbf{k},\mathbf{k}_2,-\mathbf{k}_1,\mathbf{k}_3) - V(-\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k},\mathbf{k}_3)\\
&\quad -V(-\mathbf{k}_1,\mathbf{k}_3,-\mathbf{k},\mathbf{k}_2) - V(-\mathbf{k}_1,\mathbf{k}_3,-\mathbf{k},\mathbf{k}_2)],\\
V^{(4)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) &= \tfrac{1}{3}[V(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) + V(\mathbf{k},\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_3) + V(\mathbf{k},\mathbf{k}_3,\mathbf{k}_1,\mathbf{k}_2)\\
&\quad +V(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k},\mathbf{k}_3) + V(\mathbf{k}_1,\mathbf{k}_3,\mathbf{k},\mathbf{k}_2) + V(\mathbf{k}_2,\mathbf{k}_3,\mathbf{k},\mathbf{k}_1)],
\end{aligned}$$

with $V(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = (1/8\pi)^2(\omega_k/|\mathbf{k}|)^{\frac{1}{2}}(\omega_{k1}/|\mathbf{k}_1|)^{\frac{1}{2}}(|\mathbf{k}_2|/\omega_{k2})^{\frac{1}{2}}(|\mathbf{k}_3|/\omega_{k3})^{1/2}$

$$|\mathbf{k}|\cdot|\mathbf{k}_1|[2(|\mathbf{k}|+|\mathbf{k}_1|) - |\mathbf{k}+\mathbf{k}_2| - |\mathbf{k}+\mathbf{k}_3| - |\mathbf{k}_1+\mathbf{k}_2| - |\mathbf{k}_1+\mathbf{k}_3|]. \qquad \text{(A6)}$$

The kernel $V^{(2)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ has the following symmetry property:

$$\begin{aligned}
V^{(2)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) &= V^{(2)}(\mathbf{k}_1,\mathbf{k},\mathbf{k}_2,\mathbf{k}_3) = V^{(2)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_3,\mathbf{k}_2)\\
&= V^{(2)}(\mathbf{k}_2,\mathbf{k}_3,\mathbf{k},\mathbf{k}_1). \qquad \text{(A7)}
\end{aligned}$$

We can further simplify Equation (A4) by introducing a canonical transformation, $a(\mathbf{k},t)\rightarrow b(\mathbf{k},t)$ whose coefficients $A^{(j)}$ and $B^{(j)}$ are chosen such that all the non-resonant terms ($U^{(1)}$, $U^{(3)}$, $V^{(1)}$ and $V^{(4)}$) in the Hamiltonian can be eliminated. Note, however, that the term containing $V^{(2)}$ can not be eliminated, because the following resonance conditions are involved: $\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3 = 0$ and $\omega+\omega_1-\omega_2-\omega_3 = 0$.

$$a_k = b_k + \int A^{(1)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) b_{k1} b_{k2} \delta_{k-k1-k2}\,\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2$$

$$+\int A^{(2)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)(b_{k1})^* b_{k2}\delta_{k+k1-k2}\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2$$

$$+\int A^{(3)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)(b_{k1})^*(b_{k2})^*\delta_{k+k1+k2}\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2$$

$$+\int B^{(1)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)b_{k1}b_{k2}b_{k3}\delta_{k-k1-k2-k3}\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2\,\mathrm{d}\mathbf{k}_3$$

$$+\int B^{(2)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)(b_{k1})^* b_{k2}b_{k3}\delta_{k+k1-k2-k3}\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2\,\mathrm{d}\mathbf{k}_3$$

$$+\int B^{(3)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)(b_{k1})^*(b_{k2})^* b_{k3}\delta_{k+k1+k2-k3}\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2\,\mathrm{d}\mathbf{k}_3$$

$$+\int B^{(4)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)(b_{k1})^*(b_{k2})^*(b_{k3})^*\delta_{k+k1+k2+k3}\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2\,\mathrm{d}\mathbf{k}_3, \tag{A8}$$

where

$$A^{(1)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) = -U^{(1)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)/(\omega_k-\omega_{k1}-\omega_{k2}),$$

$$A^{(2)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) = -2A^{(1)}(\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}),$$

$$A^{(3)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) = -U^{(3)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)/(\omega_k+\omega_{k1}+\omega_{k2}),$$

$$\begin{aligned}
B^{(1)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = -\{&\tfrac{2}{3}[U^{(1)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1)A^{(1)}(\mathbf{k}_2+\mathbf{k}_3,\mathbf{k}_2,\mathbf{k}_3)\\
&+U^{(1)}(\mathbf{k},\mathbf{k}_2,\mathbf{k}-\mathbf{k}_2)A^{(1)}(\mathbf{k}_1+\mathbf{k}_3,\mathbf{k}_1,\mathbf{k}_3)\\
&+U^{(1)}(\mathbf{k},\mathbf{k}_3,\mathbf{k}-\mathbf{k}_3)A^{(1)}(\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2)\\
&+U^{(1)}(\mathbf{k}_1,\mathbf{k},\mathbf{k}_1-\mathbf{k})A^{(3)}(\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_2-\mathbf{k}_3)\\
&+U^{(1)}(\mathbf{k}_2,\mathbf{k},\mathbf{k}_2-\mathbf{k})A^{(3)}(\mathbf{k}_1,\mathbf{k}_3,-\mathbf{k}_1-\mathbf{k}_3)\\
&+U^{(1)}(\mathbf{k}_3,\mathbf{k},\mathbf{k}_3-\mathbf{k})A^{(3)}(\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}_1-\mathbf{k}_2)]\\
&+V^{(1)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\}/(\omega_k-\omega_{k1}-\omega_{k2}-\omega_{k3}),
\end{aligned}$$

$$\begin{aligned}
B^{(2)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \ &A^{(1)}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1-\mathbf{k}_2)A^{(1)}(\mathbf{k}_3,\mathbf{k},\mathbf{k}_3-\mathbf{k})\\
&+A^{(1)}(\mathbf{k}_1,\mathbf{k}_3,\mathbf{k}_1-\mathbf{k}_3)A^{(1)}(\mathbf{k}_2,\mathbf{k},\mathbf{k}_2-\mathbf{k})\\
&-A^{(1)}(\mathbf{k}+\mathbf{k}_1,\mathbf{k},\mathbf{k}_1)A^{(1)}(\mathbf{k}_2+\mathbf{k}_3,\mathbf{k}_2,\mathbf{k}_3)\\
&-U^{(1)}(\mathbf{k},\mathbf{k}_2,\mathbf{k}-\mathbf{k}_2)A^{(1)}(\mathbf{k}_3,\mathbf{k}_1,\mathbf{k}_3-\mathbf{k}_1)\\
&-A^{(1)}(\mathbf{k},\mathbf{k}_3,\mathbf{k}-\mathbf{k}_3)A^{(1)}(\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2-\mathbf{k}_1)\\
&+A^{(3)}(\mathbf{k},\mathbf{k}_1,-\mathbf{k}-\mathbf{k}_1)A^{(3)}(\mathbf{k}_2,\mathbf{k}_3,-\mathbf{k}_2-\mathbf{k}_3),
\end{aligned}$$

$$
\begin{aligned}
B^{(3)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \;=\; & -\{2[U^{(1)}(\mathbf{k}_3,\mathbf{k},\mathbf{k}_3-\mathbf{k})A^{(1)}(\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2) \\
& -U^{(1)}(\mathbf{k}_1+\mathbf{k},\mathbf{k}_1,\mathbf{k})A^{(1)}(\mathbf{k}_3,\mathbf{k}_2,\mathbf{k}_3-\mathbf{k}_2) \\
& -U^{(1)}(\mathbf{k}_2+\mathbf{k},\mathbf{k}_2,\mathbf{k})A^{(1)}(\mathbf{k}_3,\mathbf{k}_1,\mathbf{k}_3-\mathbf{k}_1) \\
& -U^{(3)}(\mathbf{k}_1,\mathbf{k},-\mathbf{k}_1-\mathbf{k})A^{(1)}(\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_2-\mathbf{k}_3) \\
& -U^{(3)}(\mathbf{k}_2,\mathbf{k},-\mathbf{k}_2-\mathbf{k})A^{(1)}(\mathbf{k}_1,\mathbf{k}_3,\mathbf{k}_1-\mathbf{k}_3) \\
& +U^{(1)}(\mathbf{k},\mathbf{k}_3,\mathbf{k}-\mathbf{k}_3)A^{(3)}(\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}_1-\mathbf{k}_2)] \\
& +3V^{(1)}(\mathbf{k}_3,\mathbf{k}_2,\mathbf{k}_1,\mathbf{k})\}/(\omega_k+\omega_{k1}+\omega_{k2}-\omega_{k3}), \\
B^{(4)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \;=\; & -\{\tfrac{2}{3}[U^{(1)}(-\mathbf{k}-\mathbf{k}_1,\mathbf{k},\mathbf{k}_1)A^{(1)}(\mathbf{k}_2+\mathbf{k}_3,\mathbf{k}_2,\mathbf{k}_3) \\
& +U^{(1)}(-\mathbf{k}-\mathbf{k}_2,\mathbf{k},\mathbf{k}_2)A^{(1)}(\mathbf{k}_1+\mathbf{k}_3,\mathbf{k}_1,\mathbf{k}_3) \\
& +U^{(3)}(-\mathbf{k}-\mathbf{k}_3,\mathbf{k},\mathbf{k}_3)A^{(1)}(\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2) \\
& +U^{(3)}(\mathbf{k}+\mathbf{k}_1,\mathbf{k},\mathbf{k}_1)A^{(3)}(-\mathbf{k}_2-\mathbf{k}_3,\mathbf{k}_2,\mathbf{k}_3) \\
& +U^{(1)}(\mathbf{k}+\mathbf{k}_2,\mathbf{k},\mathbf{k}_2)A^{(3)}(-\mathbf{k}-\mathbf{k}_3,\mathbf{k},\mathbf{k}_3) \\
& +U^{(1)}(\mathbf{k}+\mathbf{k}_3,\mathbf{k},\mathbf{k}_3)A^{(3)}(-\mathbf{k}_1-\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2)] \\
& +V^{(4)}(\mathbf{k}_3,\mathbf{k}_2,\mathbf{k}_1,\mathbf{k})\}/(\omega_k+\omega_{k1}+\omega_{k2}+\omega_{k3}), \qquad \text{(A9)}
\end{aligned}
$$

the new Hamiltonian takes the form, $H(a_k, a_k^*) \to H(b_k, b_k^*)$,

$$
H = \int \omega_k b_k b_k^* \,\mathrm{d}\mathbf{k} + \frac{1}{2}\int T(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) b_k^* b_{k1}^* b_{k2} b_{k3} \delta_{k+k1-k2-k3}\,\mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2\,\mathrm{d}\mathbf{k}_3, \qquad \text{(A10)}
$$

where

$$
\begin{aligned}
T(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \;=\; & V^{(2)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \\
& -2[U^{(1)}(\mathbf{k},\mathbf{k}_2,\mathbf{k}-\mathbf{k}_2)A^{(1)}(\mathbf{k}_3,\mathbf{k}_1,\mathbf{k}_3-\mathbf{k}_1) \\
& \quad +U^{(1)}(\mathbf{k}_2,\mathbf{k},\mathbf{k}_2-\mathbf{k})A^{(1)}(\mathbf{k}_1,\mathbf{k}_3,\mathbf{k}_1-\mathbf{k}_3) \\
& \quad +U^{(1)}(\mathbf{k},\mathbf{k}_3,\mathbf{k}-\mathbf{k}_3)A^{(1)}(\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2-\mathbf{k}_1) \\
& \quad +U^{(1)}(\mathbf{k}_3,\mathbf{k},\mathbf{k}_3-\mathbf{k})A^{(1)}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1-\mathbf{k}_2) \\
& \quad -U^{(1)}(\mathbf{k}+\mathbf{k}_1,\mathbf{k},\mathbf{k}_1)A^{(1)}(\mathbf{k}_2+\mathbf{k}_3,\mathbf{k}_2,\mathbf{k}_3) \\
& \quad -U^{(3)}(-\mathbf{k}-\mathbf{k}_1,\mathbf{k},\mathbf{k}_1)A^{(3)}(-\mathbf{k}_2-\mathbf{k}_3,\mathbf{k}_2,\mathbf{k}_3)] \\
& +(\omega_k+\omega_{k1}-\omega_{k2}-\omega_{k3})B^{(2)}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3). \qquad \text{(A11)}
\end{aligned}
$$

The canonical transformation from $a_k(t)$ to $b_k(t)$, originally introduced by Zahkarov [3], was first carried out completely by Krasitskii [5]. In all the earlier work, the transformations from $a_k(t)$ to $b_k(t)$ were not canonical, and one of the direct consequences of this is that the Hamiltonian (the wave energy) in terms of the $b_k(t)$ is not conserved, see Li and Tulin [9] for the details on this issue.

The dispersion coefficients of (5) are given by

$$\begin{aligned}
\omega_0(q) &= \omega(p_0, q) = g^{\frac{1}{2}}(p_0^2 + q^2)^{\frac{1}{4}}, \\
\omega_1(q) &= \left(\frac{\partial\omega}{\partial p}\right)_0 = \tfrac{1}{2} g^{\frac{1}{2}} p_0 (p_0^2 + q^2)^{-\frac{3}{4}}, \\
\omega_2(q) &= \left(\frac{\partial^2\omega}{\partial p^2}\right)_0 = \tfrac{1}{2} g^{\frac{1}{2}} (p_0^2 + q^2)^{-\frac{3}{4}} \left[1 - \tfrac{3}{2} p_0^2/(p_0^2 + q^2)\right].
\end{aligned} \tag{A12}$$

References

1. B. Werle, Sea backscatter, spikes, and wave group observations at low grazing angles. In: IEEE International Radar Conference (1995) 187–195.
2. T. B. Benjamin and J. E. Feir, The disintegration of wave trains on deep water. *J. Fluid Mech.* 27 (1967) 417–430.
3. V. E. Zakharov, Stability of periodic waves of finite amplitude on the surface of deep fluid. *J. Appl. Mech. Tech. Phys*. 2 (1968) 190–194.
4. V. E. Zakharov, Inverse and direct cascade in the wind-driven surface waves. In: *Breaking Waves*, IUTAM Symposium Sydney, Springer (1992) 69–91.
5. V. P. Krasitskii, On reduced equations in the Hamiltonian theory of weakly nonlinear surface waves. *J. Fluid Mech.* 272 (1994) 1–20.
6. C. T. H. Baker, *The Numerical Treatment of Integral Equations*. Oxford: Clarendon Press (1977) 1030 pp.
7. G. B. Whitham, *Linear and Nonlinear Waves*. New York: John Wiley & Sons (1974) 636 pp.
8. M. P. Tulin, Breaking of ocean waves and downshifting. In: J. Grue, B. Gjevik and J.E. Weber (eds). *Waves and Nonlinear Processes in Hydrodynamics*. Dordrecht: Kluwer Academic Publishers (1996) pp. 177–190.
9. J. J. Li and M. P. Tulin, The nonlinear mechanics of gravity waves on the surface of deep water: on the nonlinear Schrödinger equation. In: M. Rahman (ed.), *Potential Fluid Flow*. Advances in Fluid Mechanics Series, Vol. 6, Computational Mechanics Publications (1995) pp. 55–108.